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Shape Preserving Interpolation Using Quadratic X-Splines

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Abstract—In this note, we use a new approach to define the Quadratic X-splines and then examine it for preserving shape when applied to strictly convex data. Here, the construction of the Quadratic X-Spline interpolation becomes straight forward.

1. INTRODUCTION

Let at the nodes

$$a = x_1 < x_2 < \cdots < x_{n+1} = b \tag{1.1}$$

the known values of the function $y(x)$ be $y_i = y(x_i)$; $i = 1, 2, \dots n+1$. We shall interpolate $y(x)$ by piecewise quadratic spline in $x \in [a, b]$. The piecewise quadratic interpolation $s(x) = s_i(x)$, on the interval $[x_i, x_{i+1}]$; $i = 1, 2, \dots, n$ with derivatives $s_i^1(x_i) = m_i$, can be written as [1]:

$$s_i(x) = a_i(x - x_i)^2 + m_i(x - x_i) + y_i; \qquad x \in [x_i, x_{i+1}], \tag{1.2}$$

where

$$a_i = \frac{(y_{i+1} - y_i - h_{i+1} m_i)}{h_{i+1}^2} \tag{1.3}$$

and

$$h_{i+1} = x_{i+1} - x_i. \tag{1.4}$$

We shall now construct the QXS interpolation by an approach different to [2] and [3]: Let $H_{o,i}(x)$ be the hermite polynomial interpolating y_i, m_i, y_{i+1}. We define in $[x_i, x_{i+1}]$

$$m_{i+1} = H_{0,i}^1(x_{i+1}); \qquad i = 1, 2, \dots, (n-1). \tag{1.5}$$

We notice that (1.5) is identical to the continuity conditions of classical quadratic spline [1].

Let $H_{1,i}(x)$ be the hermite polynomial interpolating y_i, y_{i+1}, y_{i+2} and m_i. Now set

$$m_{i+1} = H_{1,i}^1(x_{i+1}); \qquad i = 1, 2, \dots, (n-1). \tag{1.6}$$

The expanded form of (1.6) is (identical to X-spline [2]):

$$\rho_i m_i + m_{i+1} = \rho_i(3 - \rho_i)\Delta_i + (1 - \rho_i)^2 \Delta_{i+1}, \tag{1.7}$$

where

$$\Delta_i = \frac{(y_{i+1} - y_i)}{h_{i+1}}; \quad \rho_i = \frac{h_{i+2}}{(h_{i+1} + h_{i+2})}. \tag{1.8}$$

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The definition in (1.6) implies the first derivative discontinuity between $s_i(x)$ and $s_{i+1}(x)$, at the nodes $x_{i+1}; i = 1, 2, \ldots, (n-1)$. Linear combination of (1.5)–(1.6) yields:

$$m_i(\alpha_i + \beta_i \rho_i) + (\alpha_i + \beta_i) m_{i+1} = H_i(y_i, y_{i+1}, y_{i+2}); \qquad i = 1, 2, \ldots, (n-1), \tag{1.9}$$

where H_i is some known function determined from (1.5) and (1.7),(1.8). An additional information on $m_1[1]$ enables us to solve (1.9). Sufficient condition for existence of unique solution of (1.9) is [3]:

$$|\alpha_i + \rho_i \beta_i| < |\alpha_i + \beta_i|; \qquad i = 1, 2, \ldots, (n-1). \tag{1.10}$$

This also ensures that formulae (1.9) are better conditioned than the classical quadratic spline formula (1.5). X-spline formulation suggested in [2] and [3] shall also lead to (1.9); yet the following theorem is more easily proved with certain results in [2] and the Definitions (1.5) and (1.6).

THEOREM. *The quadratic X-spline defined by* (1.2)–(1.4) where the derivative parameters are determined from (1.9) and (1.10) converges as $||s - y||_\alpha = o(h^3)$.

2. SHAPE PRESERVING INTERPOLATION

Most of the shape preserving spline interpolation schemes (including those based on classical quadratic spline) either alter the derivative parameters or introduce additional nodes and construct polynomials. The present approach finds the derivatives m_i which satisfies the shape preserving criteria by selecting suitable values for (α_i, β_i). The derivatives are not modified again. When data corresponding to the nodes (1.1) is strictly convex, that is, we have [4]

$$\Delta_1 < \Delta_2 < \cdots < \Delta_n. \tag{2.1}$$

It can be easily shown that in (1.2) $s_i(x)$ is convex if

$$m_i \le \Delta_i; \qquad i = 1, 2, \ldots, n. \tag{2.2}$$

Further, $s_i(x)$ defined in $[x_i, x_{i+1}]$ to continue preserving the convex nature on to the adjacent interval $[x_{i+1}, x_{i+2}]$ at limiting point x_{i+1}, we get

$$2\Delta_i - m_i \le m_{i+1}; \qquad i = 1, 2, \ldots, (n-1). \tag{2.3}$$

Conditions (2.2),(2.3) enable (1.2) to be convex on $[a, b]$. Let us use the following notations, for $i = 1, 2, \ldots, (n-1)$:

$$\alpha_i = k_i \beta_i, \tag{2.4}$$

$$c_{i+1} = \frac{h_{i+2}}{h_{i+1}}, \tag{2.5}$$

and

$$(1 + c_{i+1}) = \frac{1}{g_{i+1}}. \tag{2.6}$$

Using (1.9), we can get

$$m_{i+1} = \frac{(2c_{i+1}\,\Delta_i\, g_{i+1} + c_{i+1}\,\Delta_i\, g_{i+1}^2 + \Delta_{i+1}\, g_{i+1}^2 - c_{i+1}\, m_i\, g_{i+1} + 2k_i\,\Delta_i - k_i\, m_i)}{(1 + k_i)};$$

$$i = 1, 2, \ldots, (n-1). \tag{2.7}$$

Find

$$Z_i = \max\left[\Delta_i - \frac{(\Delta_{i+1} - \Delta_i)}{(1 + c_{i+1})}; \quad 2\Delta_{i-1} - m_{i-1}\right] \tag{2.8}$$

and then set

$$Z_i < m_i < \Delta_i. \tag{2.9}$$

The assumption in (2.9) is consistent with (2.1). For $s_{i+1}(x)$ to be convex in $[x_{i+1}, x_{i+2}]$, we have in (2.2)

$$P_{i+1} \leq k_i R_{i+1}, \tag{2.10}$$

where

$$P_{i+1} = 2c_{i+1}\Delta_i g_{i+1} + c_{i+1}\Delta_i g_{i+1}^2 + \Delta_{i+1} g_{i+1}^2 - c_{i+1} m_i g_{i+1} - \Delta_{i+1}, \tag{2.11}$$

$$R_{i+1} = m_i + \Delta_{i+1} - 2\Delta_i. \tag{2.12}$$

From (2.9), we notice that $R_{i+1} > 0$. This implies that when

$$k_i > \tilde{k}_i = \frac{|P_{i+1}|}{R_{i+1}}, \tag{2.13}$$

the solution for $m_{i+1} < \Delta_{i+1}$ exists.

For $s_i(x)$ to preserve convexity at x_{i+1} with $s_{i+1}(x)$, we substitute m_{i+1} in (2.7) to satisfy (2.3). For this we find that for $k_i > 0$ (which is true by (2.13)) we require:

$$m_i > \left(\Delta_i - \frac{(\Delta_{i+1} - \Delta_i)}{(1 + c_{i+1})}\right) \tag{2.14}$$

This is satisfied by the setting in (2.8).

Thus, condition (2.8) has enabled $s_i(x)$ to be convex in $[x_i, x_{i+1}]$ and continue to preserve convex nature with $s_{i+1}(x)$ at x_{i+1}. This also enabled to assure the solution for $m_{i+1} < \Delta_{i+1}$. We can replace i by $(i+1)$ in (2.8) and repeat the arguments. We can thus select m_i as in (2.8) for $i = 1, 2, \ldots, (n-1)$ and preserve convexity in $[x_1, x_n]$.

However, for the last interval $[x_n, x_{n+1}]$ the condition (2.3) is absent. Here, by choosing $k_{n-1} > \tilde{k}_{n-1} = \frac{|P_n|}{R_n}$, the condition (4.12) is satisfied.

The analysis is analogous for concave data. However, in the case of monotone data, a similar advantage is not possible.

3. CONVEX APPLICATIONS

First, an example from [4], where $y(x) = 10/x^2$, and the values are known at the nodes in (1.1), $x = 1, 2, 4, 5$, and 10. We chose $m_1 = -9.5$ instead of the exact derivative value which does not satisfy the lower bound in (2.8). We set

$$m_i = \omega(\Delta_i - Z_i) + Z_i, \qquad i = 2, 3 \tag{3.1}$$

and constructed the QXS interpolation. The value of k_3 is then suitably chosen. The maximum error, $\{s(x) - y(x)\}$, at 300 equally spaced points is given in Table 1. The interval containing the points are given in Column 1 of Table 1. The sign of the error values enables to understand the nature of the QXS polynomial interpolation with respect to the actual curve. These values given in Column 2, 3, and 4 of Table 1 correspond to the choice of $\omega = 0.25, 0.5$ and 0.75 in (3.1) respectively, of which $\omega = 0.25$ is the best.

Table 1. Values of $(y(x) - s(x))$ when $y(x) = 10/x^2$.

INTERVAL	$\omega = 0.25$	$\omega = 0.5$	$\omega = 0.75$
(1.2)	−1.432	−1.432	−1.432
(2.4)	−0.308	−0.363	−0.421
(4.5)	−0.014	−0.015	−0.017
(5.10)	0.109	0.117	0.125

Table 2. Values of $(y(x) - s(x))$ when $y(x) = x^2$.

INTERVAL	$\omega = 0.25$	$\omega = 0.5$	$\omega = 0.75$
$(-2.5, -1.6)$	0.0	0.0	0.0
$(-1.6, -1)$	−0.023	−0.044	−0.060
$(-1, 0.8)$	−0.202	−0.404	−0.540
$(0.8, 1.5)$	0.046	0.049	0.112

Next, we considered the case: $y(x) = x^2$ and the nodes are $-2.5, -1.6, -1, 0.8$ and 1.5. Results are presented on similar lines as in the previous case in Table 2. We set exact value for m_i. The error values agree with estimates.

We also considered the special data [4] and found that the convex shape was preserved.

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